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Minimum steel ratios in reinforced concrete beams made of concrete with different strengths – Theoretical approach

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A B S T R A C T

Minimum reinforcement is provided in concrete beams in order to improve their behaviour towards cracking and ductility at failure.

Generally, codes of practice equations for the minimum steel ratios, longitudinal and transversal, are mainly empirical and do not include all the influential parameters in them. For this reason and due to the fact that they do lack of a theoretical background, different codes can give values for the minimum steel ratios that greatly differs from one another. Also the validity of these equations may be questioned particularly in the case of high strength concrete beams and prestressed concrete beams for which limited test data are available.

In this work, a theoretical approach for the minimum steel ratios that are required for the ductile behaviour at failure in bending, shear and torsion, in concrete beams made of concrete with different strengths is presented. Comparisons are also made between the proposed expressions, the codes expressions and available test results.

R É S U M É

Un renforcement minimum des poutres en béton armé est prévu pour améliorer leur comportement en cas de fissuration et leur ductilité dans la rupture.

En général les formules des normes techniques pour les pourcentages minimum des armatures longitudinales et transversales sont simplement empiriques et ne contiennent pas tous les paramètres influents. Pour cette raison et comme elles ne prennent pas en considération des fondements théoriques, les différents règlements donnent des valeurs très différentes pour les pourcentages minimum d'acier. La validité de ces formules peut être mise en doute particulièrement dans les cas de poutres en béton à haute résistance et des poutres précontraintes pour lesquelles seuls des résultats d'essais très limités sont disponibles.

Dans cette communication une approche théorique pour les pourcentages minimum d'acier requis pour le comportement ductile dans la rupture par flexion, effort tranchant et torsion, des poutres en béton armé exécutées avec des bétons de différentes résistances est presentée. Des comparaisons entre les expressions proposées et celles des règlements sont présentées.

1. INTRODUCTION

Design codes for reinforced and prestressed concrete structures have given lately a great deal of emphasis on the structural behaviour in the Ultimate Limit State (ULS) and beyond that state in the Post Collapse Limit State (PCLS).

In the past, the structural safety was assured only by the introduction of the partial factors of safety for both materials and loading, irrespective of the mode of failure that might occur or the post collapse residual strength. In fact, the guarantee of surplus resistance is not an overall guarantee of structural safety, since in the event of overloading and lack of member ductility at failure there will be no force redistribution and a sudden collapse without warning signs will occur.

Nowadays, the meaning of structural safety has widened up and includes the behaviour of the structure during the eventual collapse and beyond it. A structure should show enough signs (cracking and deformation) before the collapse is reached and, in the event of loosing one or few of its individual members, a progressive mode of collapse should be prevented.

For this reason and others, the minimum and maximum limits for project variables (steel ratios, member dimensions, stress level, etc.) are established in the design codes. Some of these limits, especially minimum steel ratios, have to be revised in order to cover members made of high strength concrete and prestressed members that are not addressed adequately by current codes.

Generally, minimum steel ratios are provided in reinforced concrete elements for one or more reasons: ductility and residual strength at collapse, minimum strength at ULS, minimum strength at PCLS, crack distribution and durability.

In this work, a theoretical approach for the minimum longitudinal and transversal steel ratios in beams subjected to flexure, shear and torsion, associated only with the ductility and minimum strength at ULS, is presented. The ductility through out this work is understood as the capacity of a section, a structure member or a structure as whole to undergo a reasonable amount of plastic deformation without significant loss of strength during its collapse (ULS).

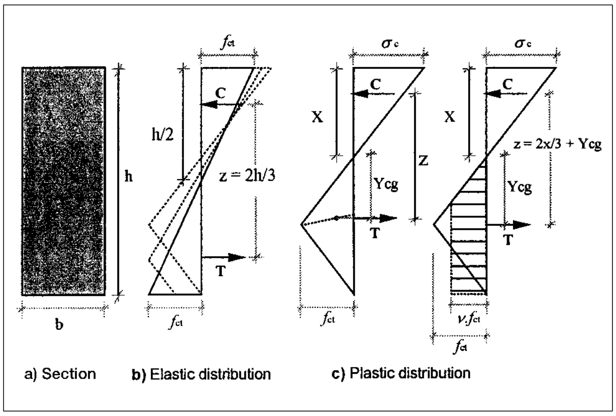

Fig. 1 – Stress distribution in a rectangular concrete section under cracking moment.

2. MINIMUM FLEXURAL STEEL RATIO IN FLEXURE

From the previous definition of ductility, the minimum flexural steel reinforcement can be any tensile longitudinal steel ratio that is smaller than the balance steel ratio, provided that the steel deformation is less than the ultimate steel deformation (deformation at steel rupture), *i.e.*:

$$\rho_{s,\min} = \frac{A_{s,\min}}{b.d} \geq \frac{M}{b.d.z \cdot f_y} = \frac{M}{b.d.(d-0.5\beta \cdot x) \cdot f_y} < \rho_{s,bal} \quad (1)$$

with

$$\rho_{s,bal} = \alpha \cdot \beta \cdot \left(\frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_y} \right) \cdot \frac{f_c}{f_y}$$

where α, β are the parameters of the rectangular stress block ($\alpha = 0.85$ and $\beta = 0.8$ for normal strength concrete).

Equation (1) is valid for any concrete strength provided that the rectangular stress block parameters (α), (β) and the ultimate concrete strain (ε_{cu}) are adjusted to the strength used. The CSA A23.3 [6], that covers concrete with strength up to 80 MPa, for example, propose $\alpha = (0.85 - 0.0015f_c) \geq 0.67$, $\beta = (0.97\text{-}0.0025f_c) \geq 0.67$, and a constant $\varepsilon_{cu} = 3.5‰$.

In order to avoid the sudden rupture of the reinforcement when the concrete reaches its ultimate strain ε_{cu}, the maximum steel deformation in the section should be limited to ultimate strain ε_{su},

$$\varepsilon_{s,\max} = \frac{d-x}{x} \cdot \varepsilon_{cu} \leq \varepsilon_{su},$$

$$\text{or} \quad \frac{x}{d} = \rho_{s,\min} \frac{f_y}{\alpha \cdot \beta \cdot f_c} = \frac{1}{\frac{\varepsilon_{su}}{\varepsilon_{cu}} + 1}$$

and, consequently:

$$\rho_{s,\min} = \alpha \cdot \beta \cdot \left(\frac{1}{\frac{\varepsilon_{su}}{\varepsilon_{cu}} + 1} \right) \cdot \frac{f_c}{f_y} \quad (2)$$

Then, the flexural minimum steel ratio that provides a minimum moment of resistance M and a certain amount of ductility at failure of a beam should satisfy both Equations (1) and (2).

The previous analysis is valid only if the beam is pre-cracked in flexure. If, on the other hand, the beam is uncracked, the following analysis should be considered taking into consideration the cracking moment M_{cr}.

In this case, it is important to guarantee that, if the cracking moment M_{cr} is reached due to eventual over-loading, the forces resisted by concrete in tension is transmitted to tensile longitudinal steel capable of resisting M_{cr}, *i.e.*,

$$\rho_{s,\min} \cdot b \cdot d \cdot f_y \cdot z = M_{cr}$$

$$\rho_{s,\min} = \frac{M_{cr}}{b \cdot d \cdot f_y \cdot z} \quad (3)$$

The value of the cracking moment M_{cr} can be obtained from the linear elastic stress distribution along the section and the tensile strength of the concrete (Fig. 1b). In the case of rectangular cross section without normal forces (ignoring shrinkage and temperature effects), the neutral axis depth is $x = 0.5\,h$ and, therefore, the cracking moment can be written as:

$$M_{cr} = \frac{f_{ct} \cdot b \cdot h^2}{6} = 0.167 f_{ct} \cdot b \cdot h^2 \quad (4a)$$

or, in the non-dimensional form,

$$\mu_{cr} = \frac{M_{cr}}{b \cdot h^2 \cdot f_c} = 0.167 \frac{f_{ct}}{f_c} \quad (4b)$$

Considering the flexural tensile strength of concrete as given by the CEB-FIP MC90 [9]:

$$f_{ct,f} = 0.2 f_{ck}^{0.67} \frac{\left(1 + 1.5\left(\frac{h}{100}\right)\right)^{0.7}}{\left(\frac{h}{100}\right)^{0.7}} \quad (5)$$

the non-dimensional cracking moment (μ_{cr}) can be expressed as a function of f_{ck}:

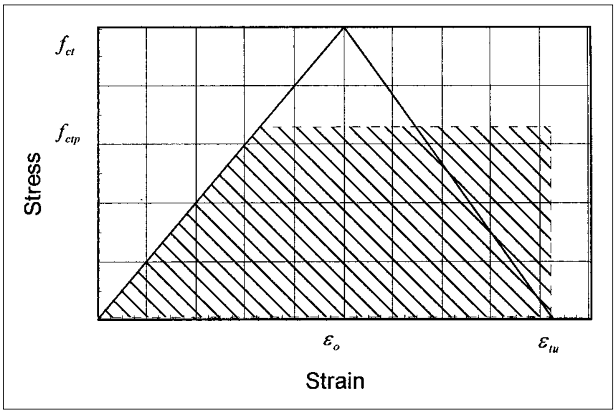

Fig. 2 – Assumed actual and plastic stress–strain curves for concrete under uniaxial tension.

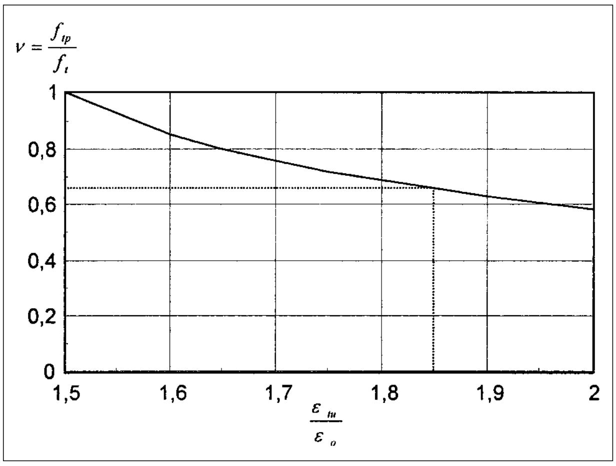

Fig. 3 – Variation of the effectiveness factor with the relationship between the maximum deformation and the deformation at peak stress for concrete in tension.

$$\mu_{cr} = 0.0333 f_{ck}^{-0.33} \frac{\left(1+1.5\left(\frac{h}{100}\right)\right)^{0.7}}{\left(\frac{h}{100}\right)^{0.7}} \qquad (6)$$

In the case of prestressed beams, M_{cr} is evaluated taking into consideration the normal force.

Admitting $z \approx 0.8h$ and $d \approx 0.85h$, from expressions (3) and (4a) one can get to:

$$\rho_{s,\min} = \frac{f_{ct,f}}{4 f_y} \qquad (7a)$$

or, as a function of f_{ck},

$$\rho_{s,\min} = 0.05 \frac{f_{ck}^{0.67}}{f_y} \frac{\left(1+1.5\left(\frac{h}{100}\right)\right)^{0.7}}{\left(\frac{h}{100}\right)^{0.7}} \qquad (7b)$$

On the other hand, if the elasto-plastic tensile stress distribution is considered in the ULS (Fig. 1c), the collapse plastic moment can be evaluated from the plastic tensile strength of concrete given as $f_{ctp} = \nu . f_{ct}$, where $\nu < 1.0$ is the effectiveness factor. The value of this factor is obtained by equating the areas underneath the stress-strain diagrams for the actual and idealised elasto-plastic diagrams (see Fig. 2), for a given relation between the ultimate deformation (ε_{tu}) and the deformation of the concrete (ε_o) at peak stress (f_{ct}). Fig. 3 gives the values of (ν) obtained as function of the ratio ($\varepsilon_{tu}/\varepsilon_o$), for a range of variation of this ratio between 1.5 and 2.0 and the stress-strain curves given in Fig. 2.

Considering that the concrete modulus of elasticity is the same for tension and compression, from the equilibrium of the section, the plastic moment at failure is:

$$M_p = \frac{1}{6} f_{ct,f} \cdot b \cdot h^2 \cdot \left(1-\frac{x}{h}\right) \cdot \left(2 \cdot \frac{x}{h} + 3 \cdot \frac{y_{cg}}{h}\right)$$

with:

$$y_{cg} = \left[1 - \frac{1}{6} \cdot \left(1 + \frac{\varepsilon_{tu}}{\varepsilon_o}\right)\right] \cdot (h-x) \ \text{ for } \ \frac{\varepsilon_{tu}}{\varepsilon_o} \le 2$$

$$\sigma_c = f_{ct,f} \cdot \frac{\varepsilon_{tu}}{\varepsilon_o} \frac{x}{h-x}$$

$$x = h \cdot \frac{\left(\sqrt{\frac{\varepsilon_{tu}}{\varepsilon_o}} - 1\right)}{\left(\frac{\varepsilon_{tu}}{\varepsilon_o} - 1\right)}$$

Assuming $\frac{\varepsilon_{tu}}{\varepsilon_o} = 1.85$, which corresponds to $\nu = 2/3$, one can obtain (see Fig. 3)

$x = 0.424\, h$

$y_{cg} = 0.302$ h (position of the resultant of the tension forces with respect to neutral axis)

$\sigma_c = 1.36 f_{ct,f}$

$M_p = 0.168 f_{ct,f} \cdot b \cdot h^2$

This moment, practically the same as the cracking moment M_{cr} given by expression (4a), leads to $\rho_{s,min}$ similar to the one obtained from expression (7b).

Fig. 4 shows comparisons made between ($\rho_{s,min} f_y$) as

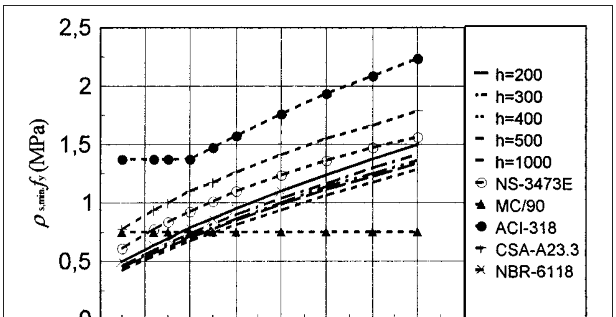

Fig. 4 – Comparison between Equation (7b) and the proposals of some codes for the longitudinal minimum steel in concrete beams (* for h >= 500 mm).

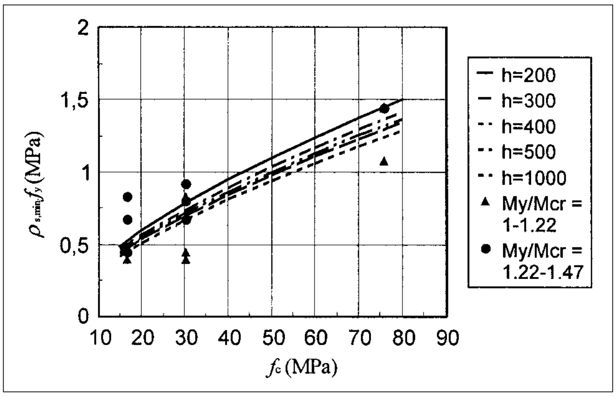

Fig. 5 – Comparison between Equation (7b) and available test results [4, 5].

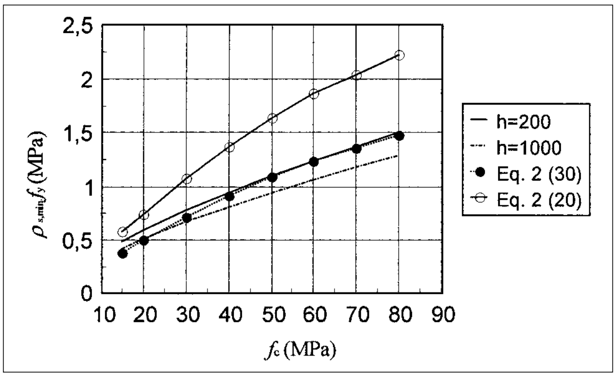

Fig. 6 – Comparison between Equation (2) ($\varepsilon_{su}/\varepsilon_{su}$ = 20 and 30) and Equation (7b) (h = 200 mm and 1000 mm).

given by expression (7b), for h varying between 200 mm and 1000 mm, and by the proposals of five codes of practice [1, 3, 6, 9 and 10]. To obtain the values of ($\rho_{s,min} f_y$) for CEB-FIP MC/90 [9] in Fig. 4, f_y = 500 MPa was used. It can be seen from this figure that expression (7b) leads to values of ($\rho_{s,min} f_y$) less than those suggested by the codes, except for the CEB-FIP MC-90, and the proposed values for the minimum steel ratios given by these codes are greatly different from one another.

Fig. 5 shows a comparison between expression (7b) and the results of the beams tested by Bosco *et al.* [4 and 5], with height (h) varying between 100 mm and 800 mm and yielding moments (M_y) close to the cracking moments. In Fig. 6 Equation (7b) is also compared with Equation (2). Two values of the ratio ($\varepsilon_{su}/\varepsilon_{cu}$) were used in Equation (2), 20 and 30, in order to represent two levels of steel ductility, medium and high. The adopted values of (α) and (β) were as proposed by the CSA 23.3 [6].

From Figs. 5 and 6 it can be concluded that expression (7b) can be used to determine the minimum longitudinal steel ratio for beams reinforced with high ductility steel ($\varepsilon_{su}/\varepsilon_{cu}$ = 30). In the case of beams reinforced with medium and low ductility steel Equation (2) becomes dominant and should be used to define the minimum reinforcement. It is worth noting that the values of ($\rho_{s,min} f_y$) given by Equation (2) with ($\varepsilon_{su}/\varepsilon_{cu}$ = 20) lie in between those given by the ACI 318-99 [1] and the NS-3474E92 [10] shown in Fig. 4.

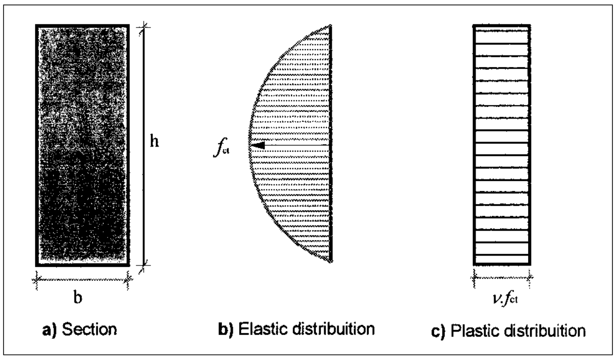

Fig. 7 – Shear stress distribution in a rectangular concrete cross–section at the instant of diagonal crack formation.

The influence of the beam height h and the concrete strength f_c on the cracking moment and on the minimum steel ratio to insure ductile behaviour of beams at failure have been investigated experimentally [4, 5] and analytically [11], but there are divergences between the obtained results. The analysis of the results of Bosco *et al.* [4 and 5] made by Queiróz [12] have indicated that the influence of h on the non-dimensional cracking moment ($M_{cr}/bh^2 f_{ct}$) is restricted to a small range of variation of (h). For practical values of h (h > ≈ 300 mm) this influence can be ignored when compared to that of f_c.

The expression for $\rho_{s,min}$ based on the fracture mechanics approach suggested by Bosco *et al.* [4 and 5], indicates a decrease in the minimum steel ratio as the beam height h increases, in the proportion of ($h^{-0,5}$). According to Ožbolt and Bruckner [11] this only happens up to a certain value of h if the beams are not provided with distributed reinforcement along their heights. In addition to that, in high beams without distributed steel, higher minimum steel ratios are required in order to avoid the sudden loss of resistance at ultimate load. Ruiz *et al.* [14], on the other hand, concluded that, besides the beam height and the concrete strength, the minimum flexural steel ratio depends also on the steel type and the concrete cover, which in part substantiate Equation (2).

3. MINIMUM SHEAR STEEL RATIO

3.1 Analysis based on the diagonal cracking load

Considering a concrete beam with rectangular cross section subjected to shear and moment, from the elastic shear stress distribution (second degree parabola, Fig. 7b), the shear crack appears when $\tau_{max} = f_{ct}$, and the associated shear force is calculated as:

$$V_{cr} = \left(\frac{2}{3}\right) \cdot b \cdot h \cdot \tau_{\max} = \left(\frac{2}{3}\right) \cdot b \cdot h \cdot f_{ct} \qquad (8)$$

When an axial force is applied to the beam, resulting in an average normal stress equal to σ_{cp} (positive when compressive), the maximum shear stress that causes the first diagonal shear crack, according to the modified Mohr-Coulomb criteria, is:

$$\tau_{\max} = \sqrt{\left(f_{ct}^2 + f_{ct} \cdot \sigma_{cp}\right)} \quad (9)$$

and the associated shear force is:

$$V_{cr} = \left(\frac{2}{3}\right) \cdot b \cdot h \cdot \sqrt{\left(f_{ct}^2 + f_{ct} \cdot \sigma_{cp}\right)} \quad (10)$$

Given that, in both cases, when the shear force V_{cr} is reached, a sudden rupture of the beam will occur with a total loss of strength, it is important that the beam contains a minimum shear reinforcement to provide it with certain ductility at failure. Considering the failure plane to be inclined at an angle θ_u to the beam axis and the stirrups at right angle to the beam axis, from the equilibrium of forces in the vertical direction along a projected length of the failure plane (*h cot* θ_u), the minimum shear reinforcement is:

$$A_{sw,\min} \cdot f_{yw} = V_{cr}$$
$$A_{sw,\min} = \rho_{sw,\min} \cdot b \cdot h \cdot \cot\theta_u \quad (11)$$

or:

$$\rho_{sw,\min} = \left(\frac{2}{3}\right) \cdot \frac{f_{ct}}{f_{yw}} \sqrt{\left(1 + \frac{\sigma_{cp}}{f_{ct}}\right)} \cdot tg\theta_u \quad (12)$$

The angle of the failure plane (θ_u) normally varies between 20° and 45°, tending to the smaller value for the minimum transversal steel ratio. Considering, then, that $\theta_u = 20°$,

$$\rho_{sw,\min} = 0{,}24 \cdot \frac{f_{ct}}{f_{yw}} \sqrt{\left(1 + \frac{\sigma_{cp}}{f_{ct}}\right)} \quad (13)$$

In the case of prestressed beams with inclined cables, the minimum transversal steel ratio can be reduced by deducting the vertical component of the prestressing force (V_p), so as:

$$A_{sw,\min} \cdot f_{yw} = V_{cr} - V_p$$

and, therefore,

$$\rho_{sw,\min} = \frac{V_{cr} - V_p}{b \cdot h \cdot f_{yw} \cdot \cot\theta_u} \geq 0 \quad (14)$$

If the plastic shear stress distribution (uniform, Fig. 7c) is used, instead of the elastic distribution, and the plastic tensile strength of the concrete is taken as $f_{ctp} = \nu . f_{ct}$, with $\nu = 2/3$, the following expression for $\rho_{sw,min}$ can be found.

$$\rho_{sw,\min} = \frac{\sqrt{\left(\nu^2 \cdot f_{ct}^2 + \nu \cdot f_{ct} \cdot \sigma_{cp}\right)}}{f_{yw} \cdot \cot\theta_u} = \frac{2}{3} \cdot \frac{f_{ct}}{f_{yw}} \sqrt{\left(1 + \frac{3\sigma_{cp}}{2 \cdot f_{ct}}\right)} \cdot tg\theta_u \quad (15)$$

Expression (15) gives the same values for $\rho_{sw,min}$ as Equation (12) when the normal stress $\sigma_{cp} = 0$, and slightly different values when $\sigma_{cp} > 0$.

This analysis is valid only for beams where the diagonal crack occurs in a region with no flexural cracks, for example in the case of fully prestressed beams. In beams with a rectangular cross section and without shear reinforcement and normal forces, test results have shown that the diagonal crack in a pre-cracked beam in flexure starts from or joins an existing flexural one, and at a shear force lesser than the one given by expression (8). The analysis of pre-cracked beams in flexure is dealt with in a later section.

3.2 Analysis based on the truss model

From the truss analogy, the stress field in the web of an uncracked concrete beam subject to flexure and shear can be considered as two orthogonal (principal stress state) uniform stress fields, one in tension and the other in compression. Considering θ and α the angles of inclination of the compression and tension fields (complementary angles), respectively, the stress in the tension field can be obtained from the truss equilibrium at the diagonal crack formation as

$$\sigma_{ct} = \frac{V_{cr}}{b \cdot z \cdot (cot\theta + cot\alpha) sin^2\alpha} = \frac{V_{cr}}{b \cdot z \cdot cot\theta} = \nu f_{ct} \quad (16)$$

From expressions (11) and (16) the cracking shear force can be found:

$$V_{cr} = \nu \cdot f_{ct} \cdot b \cdot z \cdot \cot\theta = A_{sw,\min} \cdot f_{yw} = \rho_{sw,\min} \cdot f_{yw} \cdot b \cdot h \cdot \cot\theta \quad (17)$$

Making $\nu = 2/3$ and $z \approx 0{,}8h$ in Equation (17), the minimum transversal steel ratio becomes

$$\rho_{sw,\min} = 0{,}5 \cdot \frac{f_{ct}}{f_{yw}} \quad (18)$$

Note that Equation (18), differently from Equations (13) and (15), does not take into account the influence of the normal force at the section. This occurs because in the truss analysis the top and bottom cords resist the normal forces and the stress fields of the web remains unaltered.

3.3 Minimum shear steel ratio in beams cracked in flexure

The diagonal crack formation process is very complex, and depends on the development of the flexural cracks in length and width and on the dowel effect of the flexural steel. Earlier studies have shown that the section dimensions *b* and *d*, the concrete strength and the longitudinal steel ratio influence the cracking shear force. Analyses made by Castro [8] and Queiróz [12], among others, have shown that the non-dimensional nominal cracking shear stress (V_{cr}/bdf_{ct}) decreases with the increase of f_c and *d* and increases with the increase of ρ. It was concluded also that, for beams with practical heights (*h >= 300 mm*), the influence of the effective depth of the beam *d* is not significant and, according to Regan [13], the scale effect tends to vanish in beams with some shear reinforcement.

Queiróz [12] has also shown that, in beams without

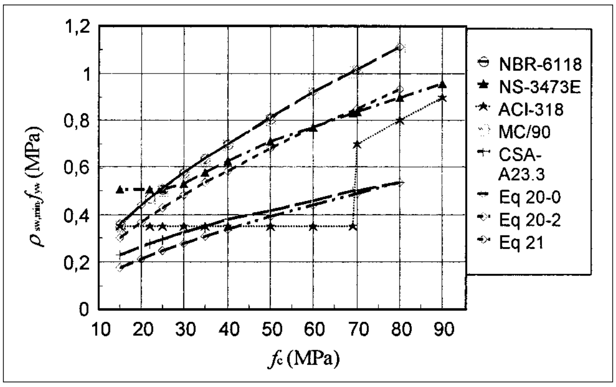

Fig. 8 – Comparisons between Equations (20) and (21) and code equations for the minimum transversal steel values.

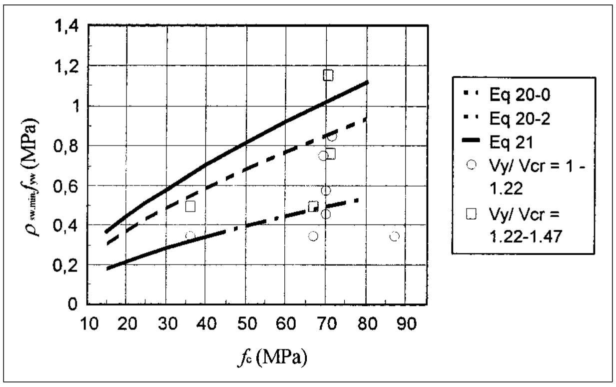

Fig. 9 – Comparison between Equations (20) and (21) and available test results.

shear reinforcement and with small flexural steel ratios (cases of beams with minimum reinforcement), the cracking shear force V_{cr} is about 40% of the one obtained from the elastic analysis.

On the basis of the previous discussion, it is here considered that the minimum shear steel ratio for a beam with bending cracking can be calculated by either Equation (13) or (18) multiplied by a reduction factor equal to 0.4 which counts for the reduction of the cracking load. Considering also that the tensile strength of the concrete (f_{ct}) is as given by CEB-FIP MC90,

$$f_{ct} = 0.3 f_{ck}^{0,67} \tag{19}$$

the minimum shear steel ratio for a beam with bending cracking can be expressed as:

$$\rho_{sw,\min} = 0.029 \cdot \frac{f_{ck}^{0,67}}{f_{yw}} \sqrt{\left(1 + \frac{\sigma_{cp}}{0.3 f_{ck}^{0,67}}\right)} \tag{20}$$

or:

$$\rho_{sw,\min} = 0.06 \cdot \frac{f_{ck}^{0,67}}{f_{yw}} \tag{21}$$

Fig. 8 compares the values of ($\rho_{sw,min} f_{yw}$) given by the expressions (20) and (21) and by those of the different codes of practice [1, 3, 6, 9 and 10], which, do not distinguish between beams with or without normal force. For the sake of comparison, Equation (20) was plotted for two values of $\sigma_{cp}/f_{ct} = \sigma_{cp}/(0.3 f_{ck}^{0.67})$: zero (curve indicated with Eq 20-0) and 2 (curve indicated with Eq 20-2). In this figure, it can be observed that Equation (21) is identical to the proposals of the new Brazilian code NBR 6118 and the CEB-FIP MC90, and leads to the highest values of ($\rho_{sw,min} f_{yw}$). Making $\sigma_{cp}/f_{ct} = 0$ in Equation (20), in general, leads to the smallest values of ($\rho_{sw,min} f_{yw}$) among all proposals.

In Fig. 9 a comparison is made between the values of ($\rho_{sw,min} f_{yw}$) of tested beams from Yoon *et al.* [15] and unpublished tests (D.Sc. thesis due for submission at COPPE/UFRJ) and the values given by Equations (20) (with $\sigma_{cp}/f_{ct} = 0$) and (21). The tested beams were divided into two groups according to the value of the relationship between the yielding shear force and the cracking shear force (V_y/V_{cr}): in the range between 1 and 1.22 and in the

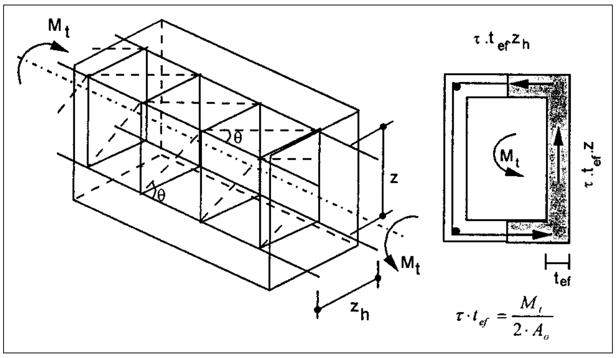

Fig. 10 – Truss model for the analysis of concrete beams under torsion.

range between 1.22 and 1.47. From this figure it seems that, at least for beams without normal force, Equation (20) can be considered to give adequate values of minimum shear steel ratio, but more test results are needed.

4. MINIMUM TORSION STEEL RATIO

Due to the little contribution of the core of solid concrete section to torsion resistance, the section core is normally ignored when beam torsion resistance is calculated. The beam section is considered as a hollow thin walled section with effective wall thickness equal to t_{ef}, as shown in Fig. 10. In design codes, t_{ef} is taken as one sixth of the short side length of the section or the relation between the section area and the section perimeter.

From the analysis of the hollow thin walls section beam under circulatory torsion, it can be obtained the shear stress in the wall under the action of the torsional moment M_t, so as:

$$\tau = \frac{M_t}{2 \cdot A_{ef} \cdot t_{ef}} \tag{22}$$

The formation of the spiralling diagonal torsion crack occurs when the shear stress in the wall reaches its limit ($\tau_{lim} = f_{ct\,p} = \nu.f_{ct}$) and, therefore, the cracking torsion moment $M_{t,cr}$ can be estimated from:

$$M_{t,cr} = 2 \cdot A_{ef} \cdot t_{ef} \cdot \nu f_{ct} \tag{23}$$

where A_{ef} is the area confined by the centre lines of the walls of the hollow section.

To guarantee the beam ductility at the instance of cracking by torsion, the beam should be provided by a minimum amount of longitudinal and transversal steel capable of resisting the forces resisted by concrete prior to cracking. This minimum steel is found by equating the forces in concrete and steel before and after cracking.

4.1 Minimum longitudinal steel ratio

From the equilibrium of forces in the space truss (Fig. 10) in the direction of the beam axis, the force in the longitudinal steel is:

$$\sum_{i=1}^{n} F_{s,l_i} = \sum_{i=1}^{n} \tau \cdot t_{ef} \cdot z_i \cdot \cot\theta - N$$

$$A_{s,\min} . f_y = \tau \cdot t_{ef} \cdot \cot\theta \cdot \sum_{i=1}^{n} z_i - N$$

where n is the number of the section walls (n = 4 for rectangular section).

The minimum longitudinal steel ratio is, then,

$$\rho_{s,\min} = \frac{\nu f_{ct}}{f_y} \cdot t_{ef} \cdot \cot\theta \cdot \frac{u_o}{A_c} - \frac{N}{A_c f_y} \quad (24)$$

where:

$$u_o = \sum_{i=1}^{n} z_i, \quad A_c = bd$$

4.2 Minimum transversal steel ratio

From the equilibrium of the truss forces in the transverse direction, the force that is transmitted to the stirrups at the time of formation of the inclined crack in the section walls is:

$$A_{sw} \cdot f_{yw} = \tau \cdot t_{ef} . z$$

and the minimum transversal steel ratio is:

$$\rho_{sw,\min} = \frac{2 A_{sw,\min}}{b \cdot z \cdot \cot\theta} = 2 \frac{\nu f_{ct}}{f_{yw}} \cdot \frac{t_{ef}}{b} \cdot tg\theta \quad (25)$$

4.3 Minimum torsion steel ratios in beams cracked in flexure

Similarly to the case of minimum shear reinforcement, it is admitted here that the diagonal-cracking load in torsion for a beam cracked in flexure is about 40% of that of the uncracked beam. Adopting the values $\nu = 2/3$ and $\theta = 20°$, from Equations (19), (24) and (25), the minimum steel ratios in torsion for a beam cracked in flexure can be deduced as:

$$\rho_{s,\min} = 0.22 \frac{f_{ck}^{0.67}}{f_y} \cdot t_{ef} \cdot \frac{u_o}{A_c} - \frac{N}{A_c f_y} \quad (26)$$

$$\rho_{sw,\min} = 0.058 \frac{f_{ck}^{0.67}}{f_{yw}} \cdot \frac{t_{ef}}{b} \quad (27)$$

Considering $u_o = 2h(0,8+0,8b/h)$, $d = 0,9h$, and $t_{ef}/b = 1/6$, $h/b = 3$ or $h/b = 2$ and $N = 0$, these equations are reduced to, respectively,

$$\rho_{s,\min} = 0.087 \frac{f_{ck}^{0.67}}{f_y} \quad \text{for} \quad h/b = 3 \quad (28a)$$

$$\rho_{s,\min} = 0.097 \frac{f_{ck}^{0.67}}{f_y} \quad \text{for} \quad h/b = 2 \quad (28b)$$

$$\rho_{sw,\min} = 0.0097 \frac{f_{ck}^{0.67}}{f_{yw}} \quad (29)$$

If, however, it is adopted for t_{ef} the relation between the area and perimeter of the section, *i.e.* $t_{ef} = 0.5b(b/h+1)$, the previous equations become:

$$\rho_{s,\min} = 0.20 \frac{f_{ck}^{0.67}}{f_y} \quad \text{for} \quad h/b = 3 \quad (30a)$$

$$\rho_{s,\min} = 0.20 \frac{f_{ck}^{0.67}}{f_y} \quad \text{for} \quad h/b = 2 \quad (30b)$$

$$\rho_{sw,\min} = 0.022 \frac{f_{ck}^{0.67}}{f_{yw}} \quad \text{for} \quad h/b = 3 \quad (31a)$$

$$\rho_{sw,\min} = 0.019 \frac{f_{ck}^{0.67}}{f_{yw}} \quad \text{for} \quad h/b = 2 \quad (31b)$$

These equations show that, depending on the adopted value of t_{ef}, the minimum steel ratios can differ considerably.

Fig. 11 shows a comparison between the values of ($\rho_{s,\min}\, f_y$) for the minimum longitudinal steel required for flexure (expression (7b), $h = 500\ mm$) and half the values required for torsion, (Equations 28a and 28b with $t_{ef} = b/6$ and $h/b = 3$ or $h/b = 2$, respectively) and Equation (30) (with $t_{ef} = 0.5b(b/h+1)$ and h/b equal to 2 or 3). Half the values for torsion were used in this comparison in order to represent the longitudinal steel required for one side of the

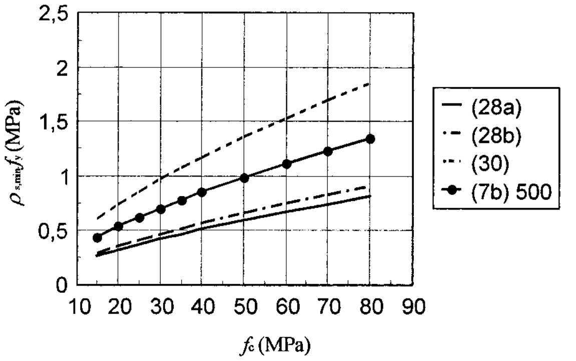

Fig. 11 – Comparison between equations for the longitudinal minimum steel for flexure (7b) and for torsion (28a , 28b and 30).

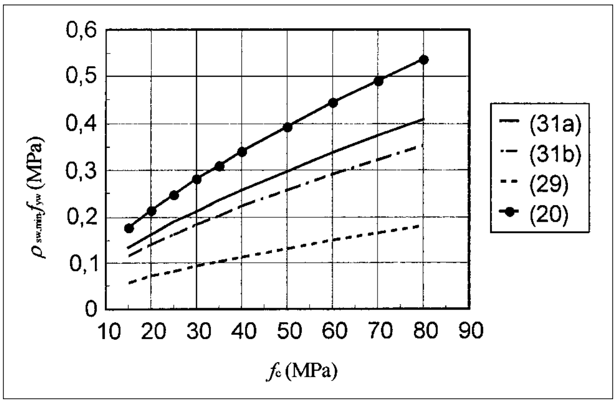

Fig. 12 – Comparison between equations for the transversal minimum steel for shear (20) and for torsion (29, 31a and 31b).

section which can be comparable to that for flexure and added to it when combined action is present.

The comparison between the values of ($\rho_{sw,min}\,f_y$) required for shear, given by expression (20), and the values of ($\rho_{s,min}\,f_y$) required for torsion moment, given by expressions (29) ($t_{ef} = b/6$), (31a) (with $t_{ef} = 0.5b(b/h+1)$ and $h/b = 3$) and (31b) (with $t_{ef} = 0.5b(b/h+1)$ and $h/b = 2$), is shown in Fig. 12.

5. CONCLUSIONS

It has been shown that the concrete strength, the steel type, the wall thickness used in torsion analysis and the normal force are the influential parameters that can greatly affect the minimum steel ratios required for flexure, shear and torsion. The beam effective depth does slightly affect the minimum steel ratio but its influence can be neglected.

There is great difference between the minimum steel ratios proposed by codes, due to the lack of theoretical background in them and the lack of enough test data that cover adequately all the influential parameters involved.

The minimum steel ratios for torsion can greatly differ from those required for flexure and shear, and, therefore, codes of practice recommendations may not be adequate.

There is need for more test results to cover a wide range of the influential parameters involved, in order to substantiate proposals for minimum steel ratios, particularly for torsion and combined actions.

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LIST OF SYMBOLS

A_c	Area of a concrete section
A_{eff}	Area confined by the centre lines of the walls of a hollow section
A_s	Area of longitudinal steel
$A_{s,min}$	Minimum area of longitudinal steel
A_{sw}	Area of transversal steel
$A_{sw,min}$	Minimum area of transversal steel
$F_{s,l}$	Force in longitudinal steel
M	Bending moment
M_{cr}	Cracking bending moment
M_p	Plastic flexural moment
M_t	Torsional moment
N	Normal force
V_{cr}	Cracking shear force
V_p	Shear force due to the inclination of prestressing cables
b	Width of a rectangular section
d	Effective depth of a rectangular section
h	Section height
t_{ef}	Effective thickness of a thin walled section
x	Neutral axis depth
z	Lever arm
f_c	Compressive strength of concrete

f_{ck} Characteristic compressive strength of concrete
f_{ct} Tensile strength of concrete
$f_{ct,f}$ Flexural tensile strength of concrete
f_y Yield strength of steel
f_{yw} Yield strength of web steel
n Number
α Coefficient
β Coefficient
θ_u Truss angle
ε_{co} Concrete strain at peak stress in compression
ε_{cu} Ultimate concrete strain in compression
ε_{su} Ultimate steel strain
ε_{tu} Ultimate concrete strain in tension
ε_o Concrete strain at peak stress in tension
σ_{cp} Average stress in concrete section due to normal force
τ_{max} Maximum shear stress
μ_{cr} Normalized cracking moment (Mcr/fc.b.h2)
$\rho_{s,bal}$ Balanced steel ratio
$\rho_{s,min}$ Minimum longitudinal steel ratio
$\rho_{sw,min}$ Minimum transversal steel ratio
ν Effectiveness factor